CRUISE CONTROLLER DESIGN AND TRAFFIC FLUIDS

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Our team

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Our team

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Markos is the Master Mind behind the project!

A New Science

THE SCIENCE OF AUTOMATED VEHICLE TRAFFIC

Foundations built with the help of

Theory of PDEs
Numerical Analysis
Fluid mechanics
Dynamical Systems
Mathematical Physics

A New Science

THE SCIENCE OF AUTOMATED VEHICLE TRAFFIC

but most of all

NONLINEAR CONTROL THEORY

The Science of Conventional Traffic

Continuity Equation	$\rho_t + \left(\rho v\right)_x = 0$
Reduced Model	LWR: $v = f(\rho)$, $\rho > 0, v > 0$
Velocity Equation	ARZ: $v_t + \left(v + \rho f'(\rho)\right) v_x$ $= -k\left(v - f(\rho)\right)$ $\rho > 0, v \in \mathbb{R}$
Derived equation	$s_t + v s_x = -ks$, $s = v - f(\rho)$
Constants	$k > 0$
Functions	$f : (0, +\infty) \to (0, +\infty)$ decreasing

The Science of Conventional Traffic

Anisotropy → Follow-the-Leader Models

n identical particles of mass $1/n$, moving on a straight line

$$\dot{x}_i = v_i$$

$$\dot{v}_i = \frac{n(v_{i-1} - v_i)}{\tau(ns_i)} - k\left(v_i - f\left(\frac{1}{ns_i}\right)\right)$$

$$f(\rho) = \int_0^{1/\rho} \frac{ds}{\tau(s)} \qquad s_i = x_{i-1} - x_i$$

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$$\dot{x}_i = f\left(\frac{1}{ns_i}\right)$$

Questions

What will be the equations for automated vehicles?

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What will happen if we allow lane-free movement?

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What will happen if we allow lane-free movement?

What will happen if we allow nudging?

Idea

Lane-free movement and nudging?

This reminds us something…

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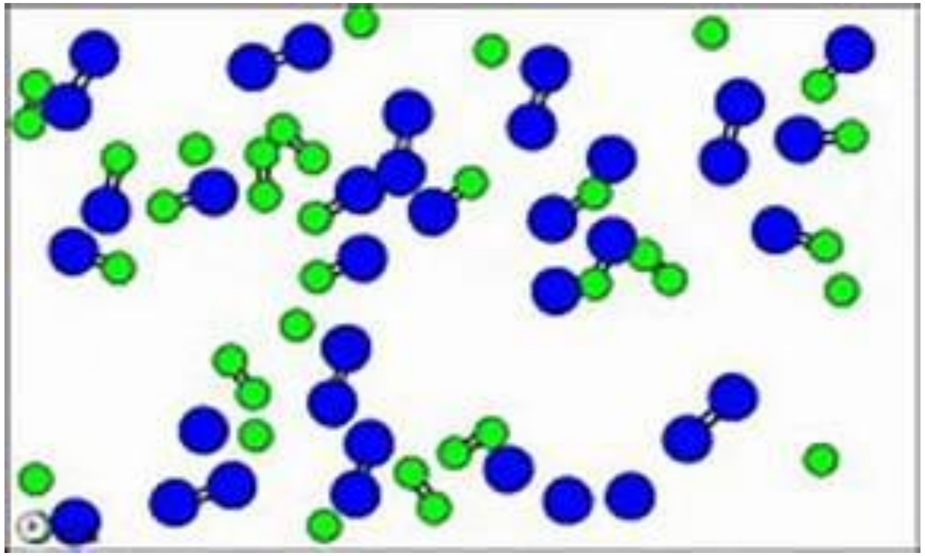


It's like having molecules of a fluid!!

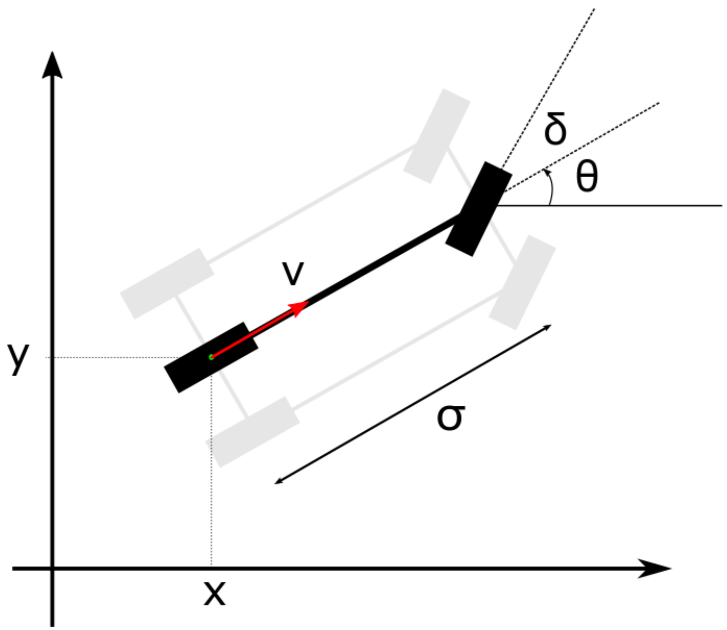
But fluid equations are isotropic!!

Conclusion:

We have to think everything from scratch…

Let’s start from the basics

The bicycle kinematic model



$$\dot{x}_i = v_i \cos(\theta_i)$$

$$\dot{y}_i = v_i \sin(\theta_i)$$

$$\dot{\theta}_i = \sigma_i^{-1} v_i \tan(\delta_i)$$

$$\dot{v}_i = F_i$$

A simplification

$$\delta_i = \arctan\left(\frac{\sigma_i u_i}{v_i}\right), \; i = 1, \ldots, n$$

$$\boxed{\begin{aligned} \dot{x}_i &= v_i \cos(\theta_i) \\ \dot{y}_i &= v_i \sin(\theta_i) \\ \dot{\theta}_i &= u_i \\ \dot{v}_i &= F_i \end{aligned}}$$

for $i = 1, \ldots, n$, where u_i and F_i are the inputs of the system

What do we really want?

Let's assume the simplest road

$$\left\{ (x, y) \in \mathbb{R}^2 : |y| < a \right\}$$

Distance between vehicles

$$d_{i,j} := \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}\text{, for } i, j = 1, \ldots, n$$

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4) The orientation angles of the vehicles should be bounded

$$\left| \theta_i\left(t\right) \right| < \varphi < \frac{\pi}{2}$$

What do we really want?

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The 6th constraint is different: input constraint

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$$\left|F_i\left(t\right)\right| \leq F_{\max}$$

The 6th constraint is different: input constraint

7) If possible: $\lim_{t\to+\infty}\left(v_i(t)\right) = v^*,\ \lim_{t\to+\infty}\left(\theta_i(t)\right) = \lim_{t\to+\infty}\left(u_i(t)\right) = \lim_{t\to+\infty}\left(F_i(t)\right) = 0$

The state space

The state: $w = (x_1, \ldots, x_n, y_1, \ldots, y_n, \theta_1, \ldots, \theta_n, v_1, \ldots., v_n) \in \mathbb{R}^{4n}$

The state space:

$$\Omega := \left\{ w \in \mathbb{R}^{4n} : \begin{array}{c} x_i \in \mathbb{R}, |y_i| < a, i = 1, \ldots, n \\ v_i \in (0, v_{\max}), |\theta_i| < \varphi, i = 1, \ldots, n \\ d_{i,j} > L_{i,j}, i, j = 1, \ldots, n, j \neq i \end{array} \right\}$$

An open set (not diffeomorphic to $\mathbb{R}^{4n}$)

What do we really want?

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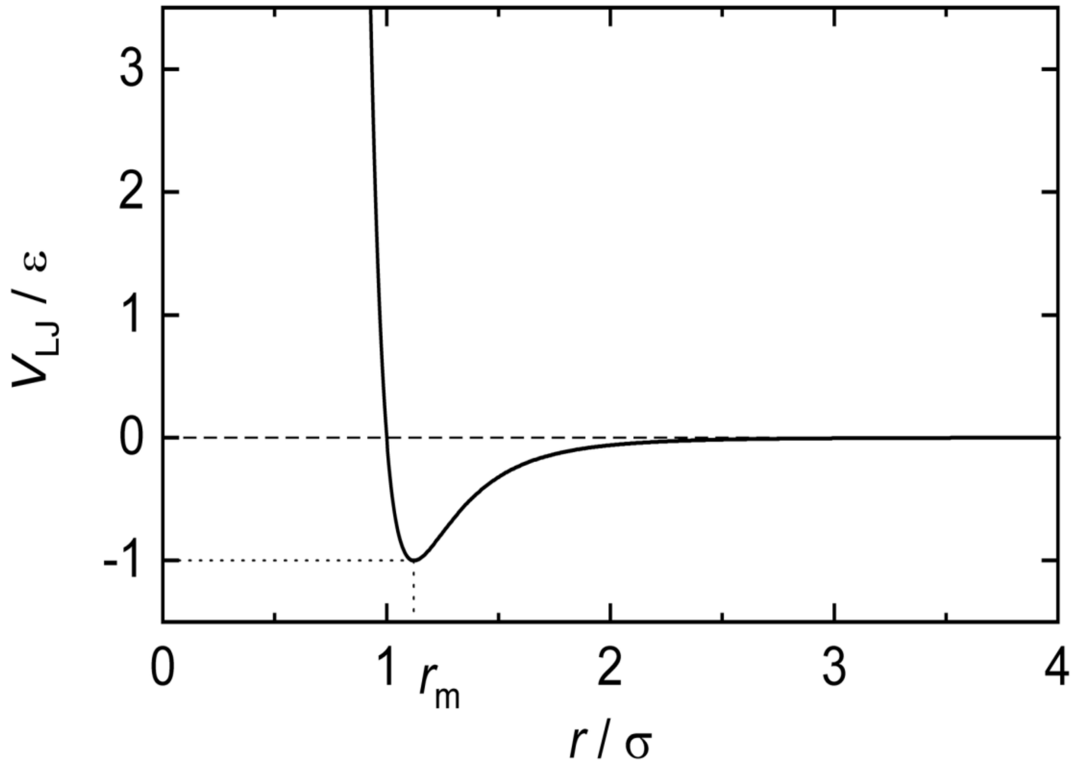
State space being an open set and a non-compact set of equilibria?

A case rarely studied in

NONLINEAR CONTROL THEORY

How nature does the job?

By means of potential functions!!



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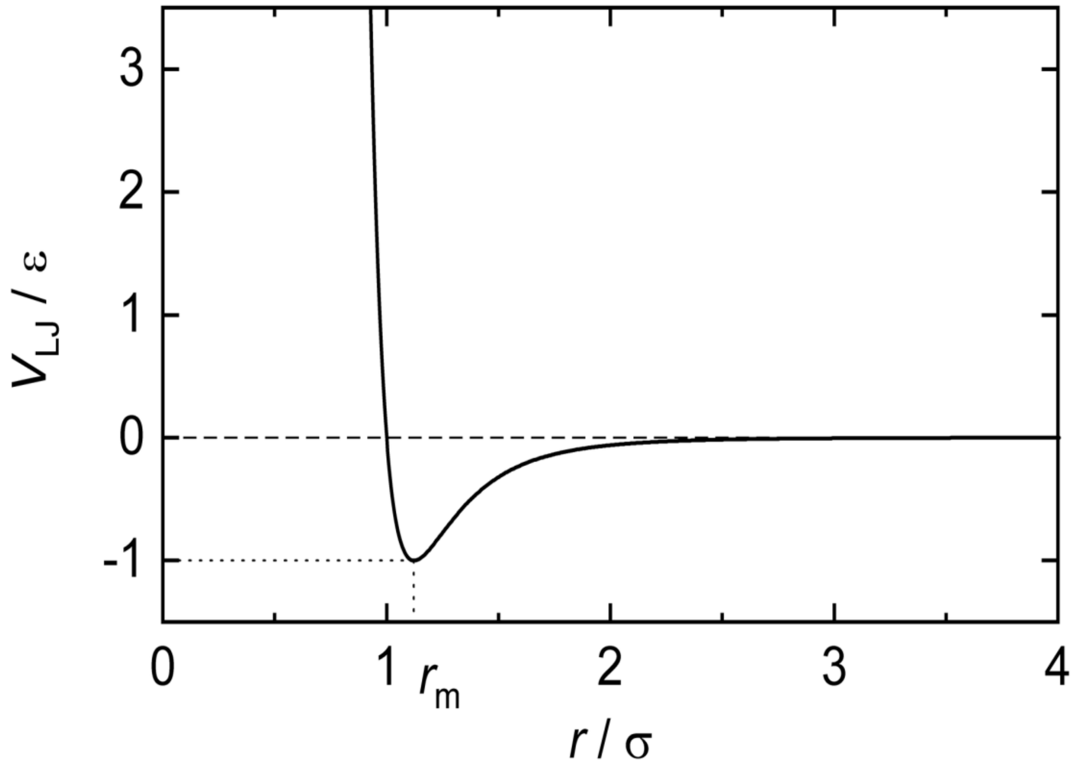
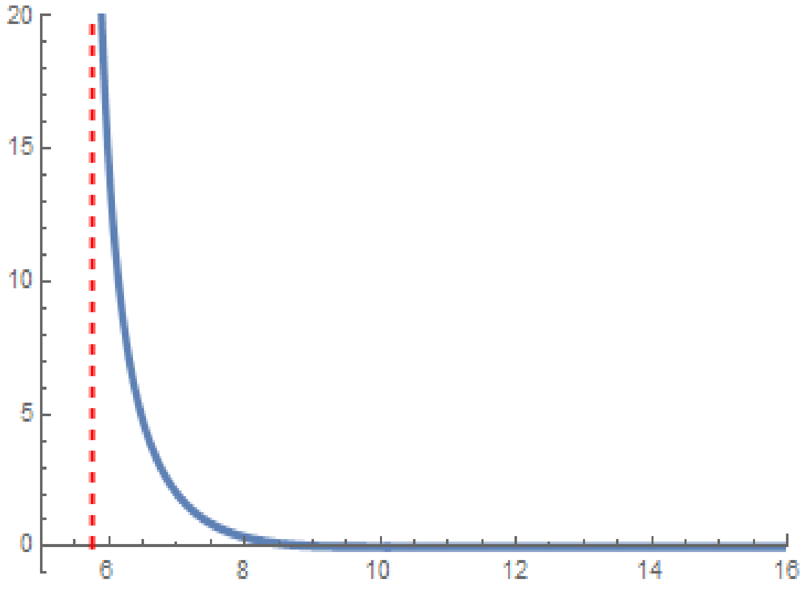


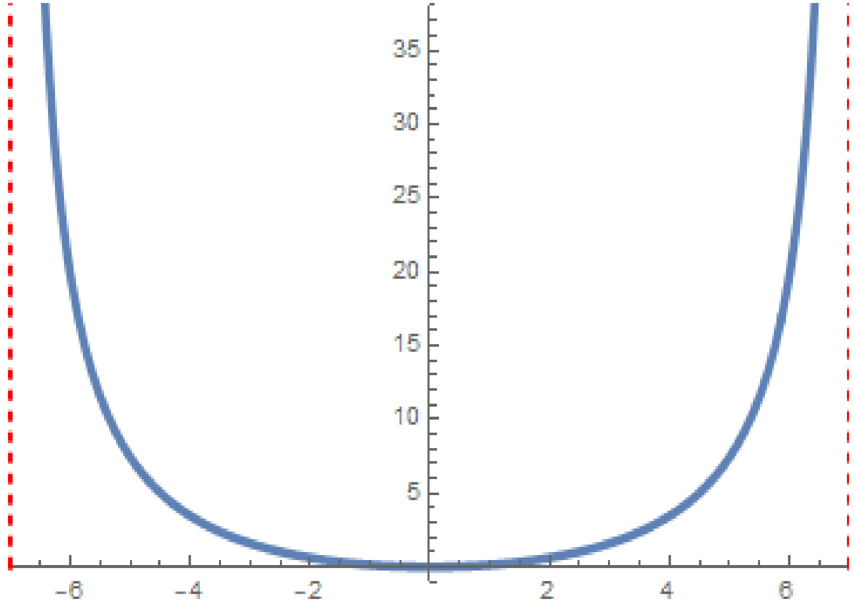
So, let's imitate nature!

Idea!

Think of vehicles as "particles"-"molecules" (n-body problem)

Use potential functions to avoid collisions and escape from the road





$\lambda > L_{i,j} \rightarrow$ interaction distance

Use some kind of "mechanical energy" of the "particles" as a Lyapunov function

Lyapunov Function

$$H(w) = \sum_{i=1}^{n} U_i\left(y_i\right) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i} V_{i,j}\left(d_{i,j}\right)$$ → Potential energy

$$+\frac{1}{2}\sum_{i=1}^{n}\frac{\left(v_i\cos(\theta_i)-v^*\right)^2+bv_i^2\sin^2(\theta_i)}{v_i\left(v_{\max}-v_i\right)}$$ → "Relativistic" kinetic energy

$$+A\sum_{i=1}^{n}\left(\frac{1}{\cos(\theta_i)-\cos(\varphi)}-\frac{1}{1-\cos(\varphi)}\right)$$ → Penalty term

When $w \to \partial\Omega$ then $H(w) \to +\infty$

The PRCC

$$F_i = -\frac{1}{q(v_i, \theta_i)} f\left(v_i \cos(\theta_i) - v^*\right)$$

→ Friction term/relaxation term

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$$-\frac{1}{q\left(v_i, \theta_i\right)} \sum_{j \neq i} V'_{i,j}\left(d_{i,j}\right) \frac{x_i - x_j}{d_{i,j}}$$

→ Repulsion term/pressure term

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$$-\frac{1}{q(v_i,\theta_i)} \sum_{j \neq i} \kappa\left(d_{i,j}\right)\left(v_i \cos(\theta_i) - v_j \cos(\theta_j)\right)$$ → Viscosity term

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$$F_i = -\frac{1}{q\left(v_i, \theta_i\right)} f\left(v_i \cos(\theta_i) - v^*\right)$$ → Friction term/relaxation term

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$$-\frac{1}{q\left(v_i, \theta_i\right)} \sum_{j \neq i} \kappa\left(d_{i,j}\right)\left(v_i \cos(\theta_i) - v_j \cos(\theta_j)\right)$$ → Viscosity term

Moreover, there exists $R \in K_\infty$ such that $\left|F_i\right| \leq R\left(H(w)\right)$.

The PRCC

$$\nabla H(w)\dot{w} = -\sum_{i=1}^{n}\left(v_i\cos(\theta_i) - v^*\right) f\left(v_i\cos(\theta_i) - v^*\right)$$

$$-\sum_{i=1}^{n} v_i\sin(\theta_i) f\left(v_i\sin(\theta_i)\right)$$

$$-\frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}\kappa\left(d_{i,j}\right)\left(v_i\cos(\theta_i) - v_j\cos(\theta_j)\right)^2$$

$$-\frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}\kappa\left(d_{i,j}\right)\left(v_i\sin(\theta_i) - v_j\sin(\theta_j)\right)^2$$

Barbălat's lemma → $\lim_{t\to+\infty}\left(v_i(t)\right) = v^*,\ \lim_{t\to+\infty}\left(\theta_i(t)\right) = \lim_{t\to+\infty}\left(u_i(t)\right) = \lim_{t\to+\infty}\left(F_i(t)\right) = 0$

$$\left|F_i(t)\right| \leq R\left(H(w(0))\right) \leq F_{\max}$$

What have we done?

n identical particles of mass $1/n$, same potentials, same distance metrics, moving on a straight line

$$
\begin{aligned}
s_i &= x_{i-1} - x_i \\
\dot{x}_i &= v_i \\
q(v_i)\dot{v}_i &= -f\left(v_i - v^*\right) \\
&\quad + n\left(V'\left(ns_i\right) - V'\left(ns_{i+1}\right)\right) \\
&\quad + n^2\left(\kappa\left(ns_i\right)(v_{i-1} - v_i) - \kappa\left(ns_{i+1}\right)(v_i - v_{i+1})\right)
\end{aligned}
$$

$$
q(v) = \frac{v_{\max}^3 (v + v^*) - 2v_{\max}^2 v^* v}{2\left(v_{\max} - v\right)^2 v^2}
$$

ISOTROPY!!—LIKE REAL FLUIDS!!

What have we done?

Continuity equation: $\rho_t + (\rho v)_x = 0$

Velocity equation: $q(v)(v_t + vv_x) + \rho^{-1}P'(\rho)\rho_x = \rho^{-1}(\mu(\rho)v_x)_x - f(v - v^*)$

Navier-Stokes: $v_t + vv_x + \rho^{-1}P'(\rho)\rho_x = \rho^{-1}(\mu(\rho)v_x)_x - \gamma(\rho, v)v$

Differences:

$q(v)$ → the effect of "relativistic" kinetic energy; appears in relativistic fluid mechanics

Friction terms: $-f(v - v^*)$ instead of $-\gamma(\rho, v)v$

What have we done?

We can design our own fluid!!

$$P(\rho) = z - V'\left(\frac{1}{\rho}\right),\ \rho \in \left(0, \rho_{\max}\right)$$

$$\mu(\rho) = \frac{1}{\rho}\kappa\left(\frac{1}{\rho}\right),\ \rho \in \left(0, \rho_{\max}\right)$$

$$\rho_{\max} = \frac{1}{L}$$

$\mu(\rho) = P'(\rho) = 0$ for $0 < \rho \le \bar{\rho} = \frac{1}{\lambda},\ \lim_{\rho \to \rho_{\max}^{-}} \left(P(\rho)\right) = +\infty$

What have we done?

	Human Drivers	Automated Vehicles
Continuity Equation	$\rho_t + (\rho v)_x = 0$	
Reduced Model	LWR: $v = f(\rho)$, $\rho > 0, v > 0$	$v = \beta^{-1}\left(-\rho^{-2}\mu(\rho)\rho_x\right)$, $\rho \in (0, \rho_{\max}), v \in (0, v_{\max})$
Velocity Equation	ARZ: $v_t + \left(v + \rho f'(\rho)\right)v_x = -k\left(v - f(\rho)\right)$ $\rho > 0, v \in \mathbb{R}$	$\beta'(v)\left(v_t + vv_x\right) + k\rho^{-2}\mu(\rho)\rho_x = \rho^{-1}\left(\mu(\rho)v_x\right)_x - k\beta(v)$ $\rho \in (0, \rho_{\max}), v \in (0, v_{\max})$
Derived equation	$s_t + vs_x = -ks$, $s = v - f(\rho)$	$s_t + vs_x = -ks$, $s = \beta(v) + \rho^{-2}\mu(\rho)\rho_x$

More material…

I. K., D. Theodosis and M. Papageorgiou, “Constructing Artificial Traffic Fluids by Designing Cruise Controllers”, *Systems & Control Letters,* 167 (2022), 105317.

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NEWTONIAN CONTROLLERS IN:

I. K., D. Theodosis and M. Papageorgiou, “Lyapunov-based Two-Dimensional Cruise Control of Autonomous Vehicles on Lane-Free Roads”, *Automatica,* 145 (2022), 110517.

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I. K., D. Theodosis and M. Papageorgiou, “Constructing Artificial Traffic Fluids by Designing Cruise Controllers”, *Systems & Control Letters,* 167 (2022), 105317.

NEWTONIAN CONTROLLERS IN:

I. K., D. Theodosis and M. Papageorgiou, “Lyapunov-based Two-Dimensional Cruise Control of Autonomous Vehicles on Lane-Free Roads”, *Automatica,* 145 (2022), 110517.

RING ROADS OF CONSTANT WIDTH:

D. Theodosis, I. K. and M. Papageorgiou, “Cruise Controllers for Lane-Free Ring-Roads Based on Control Lyapunov Functions”, *Journal of the Franklin Institute*, 360 (2023), 6131-6166.

More material…

NUMERICAL SCHEMES FOR THE NONLINEAR HEAT EQUATION

$$\rho_t + \left(\rho\beta^{-1}\left(-\rho^{-2}\mu(\rho)\rho_x\right)\right)_x = 0$$

D. Theodosis, I. K., G. Titakis, I. Papamichail and M. Papageorgiou, “A nonlinear heat equation arising from automated-vehicle traffic flow models”, *Journal of Computational and Applied Mathematics*, 437 (2024), 115443.

More material…

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WHAT KIND OF STABILITY DO WE HAVE?

I. K., D. Theodosis and M. Papageorgiou, "Stability analysis of nonlinear inviscid microscopic and macroscopic traffic flow models of bidirectional cruise-controlled vehicles", *IMA Journal of Mathematical Control and Information,* 39 (2022), 609-642.

More material…

The case of variable-width roads with possible on-ramps and off-ramps in

I. K., D. Theodosis and M. Papageorgiou, “Forward Completeness and Applications to Control of Automated Vehicles”, submitted to *IEEE TAC* (see also arXiv:2307.11515 [math.OC]).

each vehicle moves within its own (curved) corridor

each vehicle has its own desired speed + all features of the controllers in the constant-width case

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each vehicle moves within its own (curved) corridor

each vehicle has its own desired speed + all features of the controllers in the constant-width case

boundaries are required to be known only in a neighborhood of the vehicle

$\Downarrow$

obstacle avoidance,

cases where vehicles with a large (or small) desired speed may prefer (or be restricted) to be in the left (or right) side of the road.

More material…

IS THE DERIVATION RIGOROUS?

Very deep question, related to Hilbert's 6th problem (currently unsolved)

A similar problem

n identical NEWTONIAN particles of mass $1/n$, same potentials, moving on a straight line

$$\begin{aligned}
s_i &= x_{i-1} - x_i \\
\dot{x}_i &= v_i \\
\dot{v}_i &= n\left(V'\left(ns_i\right) - V'\left(ns_{i+1}\right)\right) \\
&\quad + n^2\left(\kappa\left(ns_i\right)(v_{i-1} - v_i) - \kappa\left(ns_{i+1}\right)(v_i - v_{i+1})\right)
\end{aligned}$$

$\kappa(s) > 0$ for all $s > 0$

More material…

Do we get a (weak) solution of the 1-D compressible Navier-Stokes equations

$$\begin{aligned} &\rho_t + (\rho v)_x = 0 \\ &v_t + v v_x + \rho^{-1} P'(\rho)\rho_x = \rho^{-1}\left(\mu(\rho) v_x\right)_x \end{aligned}$$

as $n \to +\infty$ with $P(\rho) = z - V'\left(\frac{1}{\rho}\right)$, $\mu(\rho) = \frac{1}{\rho}\kappa\left(\frac{1}{\rho}\right)$ for $\rho > 0$?

More material…

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$$\rho_t + (\rho v)_x = 0$$
$$v_t + v v_x + \rho^{-1} P'(\rho)\rho_x = \rho^{-1}\left(\mu(\rho) v_x\right)_x$$

as $n \to +\infty$ with $P(\rho) = z - V'\left(\frac{1}{\rho}\right)$, $\mu(\rho) = \frac{1}{\rho}\kappa\left(\frac{1}{\rho}\right)$ for $\rho > 0$?

ANSWER: YES!!!

I. K. and M. Papageorgiou, "A particle method for 1-D compressible fluid flow", to appear in *Studies in Applied Mathematics* (see also arXiv:2301.04553 [math.AP]).

Just some of the foundations of the new science

Many more to be done…

THANK YOU!